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BOUNDARY CONTROL OF QUASILINEAR ELLIPTIC EQUATIONS

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BOUNDARY CONTROL OF QUASILINEAR ELLIPTIC EQUATIONS
CONTROLE FRONTIERE D'EQUATIONS QUASI-LINEAIRES ELLIPTIQUES

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ABSTRACT

In this work, we study some boundary control problems of systems governed by quasilinear elliptic equations. Our main interest consists in the derivation of optimality conditions. The fundamental difficulty resides in the differentiability of the state respect to the control. We overcome it distinguishing two cases, depending on the polynomial growth order α of operator coefficients: in case $\alpha \geqq 2$, we prove the previous differentiability in the Gâteaux sense, introducing some function spaces (naturally associated with the state equation); in case $1<\alpha<2$, we approximate the initial control problem by a family of problems corresponding to the case $\alpha \geqq 2$.

RESUME

Dans ce travail nous étudions quelques problèmes de contrôle frontière de systèmes gouvernés par des équations elliptiques quasilinéaires. Notre principal but est la dérivation des conditions d'optimalité. La difficulté fondamentale réside en la différentiabilité de l'état par rapport au contrôle. Pour surmonter cette difficulté nous distinguons deux cas dépendant de l'ordre de croissance α des coefficients de l'opérateur: dans le cas $\alpha \geqq 2$ nous montrons la différentiabilité dans le sens Gâteaux introduisant quelques espaces de fonctions (naturellement associés à l'équation d'état); dans le cas $1<\alpha<2$ nous approchons le problème de contrôle initial par une famille de problèmes correspondant au cas $\alpha \geqq 2$.

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1.- INTRODUCTION

This paper deals with boundary control problems of systems governed by quasilinear strongly elliptic equations. Our aim is to derive the optimality system.

Let Ω be an open and bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary Γ (Nečas (17)). Let us consider the following differential operator

$$Ay = -\mathrm{div}\ (a(x,\nabla y)) + a_0(x,y) \tag{1.1}$$

where

$$a(x,\eta) = (a_1(x,\eta),\ldots,a_N(x,\eta))$$

We will assume the conditions

$$\left.\begin{array}{l} a_j(.,\eta) \text{ is a measurable function on } \Omega \\ a_j(x,.) \text{ belongs to } C^1(\mathbb{R}^N) \quad j=1,..,N \end{array}\right. \tag{1.2}$$

$$\left.\begin{array}{l} a_0(.,s) \text{ is a measurable function on } \Omega \\ a_0(x,.) \text{ belongs to } C^1(\mathbb{R}) \end{array}\right. \tag{1.3}$$

$$\sum_{i,j=1}^{N} \frac{\partial a_j}{\partial \eta_i}(x,\eta)\xi_i\xi_j \geq \Lambda_1(1+|\eta|)^{\alpha-2}|\xi|^2 \tag{1.4}$$

$$\sum_{i,j=1}^{N} \left|\frac{\partial a_j}{\partial \eta_i}(x,\eta)\right| \leq \Lambda_2(1+|\eta|)^{\alpha-2} \tag{1.5}$$

$$\Lambda_3 < \frac{\partial a_0}{\partial s}(x,s) \leq \Lambda_4(1+|s|)^{\alpha-2} \tag{1.6}$$

$$a_0(x,0) = a_j(x,0) = 0 \quad j=1,\ldots,N \tag{1.7}$$

for some $\alpha\varepsilon(1,+\infty)$, some strictly positive constants $\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4$ all $x\varepsilon\Omega$, all $s\varepsilon\mathbb{R}$ and all $\eta,\xi\varepsilon\mathbb{R}^N$.

Remark. Hypotheses (1.6) and (1.7) can be relaxed in the following form:

There exist positive constants Λ_3 and Λ_4 and a medible subset Ω' such that $m(\Omega')>0$ and

$$0<\frac{\partial a_0}{\partial s}(x,s) \leqq \Lambda_4(1+|s|)^{\alpha-2} \qquad \forall(x,s)\varepsilon\Omega x\mathbb{R} \tag{1.6'}$$

$$\frac{\partial a_0}{\partial s}(x,s) > \Lambda_3 \qquad \forall(x,s)\varepsilon\Omega' x\mathbb{R}$$

$$a_0(.,0)-\operatorname{div}(a(.,0))\varepsilon(W^{1,\alpha}(\Omega))' \tag{1.7'}$$

In this case, it is enough to do the change

$$\tilde{a}_0(x,s) = a_0(x,s)-a_0(x,0) \qquad \text{and}$$

$$\tilde{a}(x,\eta) = a(x,\eta)-a(x,0)$$

We are concerned with the following control system

$$\begin{cases} Ay = f & \text{in } \Omega \\ \dfrac{\partial y}{\partial n_A} = u & \text{on } \Gamma \end{cases} \tag{1.8}$$

where $\frac{\partial y}{\partial n_A}(x) = \sum_{j=1}^{N} a_j(x,\nabla y(x))n_j(x)$ and $\vec{n}(x)=(n_1(x),\dots,n_N(x))$ is the unit outward normal vector to Γ at x.

In the sequel, let us denote by $(\ ,\ )_\Omega$ the duality product between $(W^{1,\alpha}(\Omega))'$ and $W^{1,\alpha}(\Omega)$, the usual Sobolev space. Also, we consider (Kufner et al. (12))

$$W^{1/\beta,\alpha}(\Gamma) = \{u|_\Gamma : u\varepsilon W^{1,\alpha}(\Omega)\}$$

endowed with the usual norm

$$||h||_{W^{1/\beta,\alpha}(\Gamma)} = \operatorname{Inf}\{||u||_{W^{1,\alpha}(\Omega)} : u|_\Gamma = h\}$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $u|_\Gamma$ is the trace of u on Γ.

Let $W^{-1/\beta,\beta}(\Gamma)$ denote the dual space of $W^{1/\beta,\alpha}(\Gamma)$ and $(\ ,\ )_\Gamma$ their duality product. It is well known (J.L. Lions (13)) that problem (1.8) has a unique weak solution $y_u \varepsilon W^{1,\alpha}(\Omega)$ for each $f\varepsilon(W^{1,\alpha}(\Omega))'$ and $u\varepsilon W^{-1/\beta,\beta}(\Gamma)$.

In case $f\varepsilon L^\beta(\Omega)$, we may give a rigorous interpretation of the boundary condition satisfied by the weak solution (see Casas and Fernández (5)).

There are many works dealing with control problems governed by linear PDE, we mention specially here J.L. Lions (14). Control of semilinear equations has been also studied by some authors J.L. Lions (15), Komornik (10), Bonnans and Casas (3),(4). However, we do not know other results about the control of quasilinear elliptic equations, except our recent work (6).

Quasilinear case introduces respect to other cases, the fundamental difficulty of the differentiability of the state respect to the control. The same difficulty also appears in optimal control of variational inequalities and two different methods have been used to overcome it. The first one consists in proving differentiability in a certain sense (Mignot (16)) and the second idea is to approximate the given problem by a family of smooth problems and then to tend to the limit in the corresponding optimality conditions (Barbu (2)).

In this work, we use both methods. In fact, we will distinguish two cases, depending on the polynomial growth order α of operator coefficients.

The plan of this paper is the following: in section 2 we study the state equation and when $\alpha \geqq 2$ we prove the differentiability of state respect to the control by introducing some function spaces; in sections 3 and 4 we formulate the control problem and we derive the optimality system in case $\alpha \geqq 2$ and $1<\alpha<2$ respectly.

Finally, let us give two examples of the principal part of operators A that satisfy the previous hypotheses:

EXAMPLE 1

$$a(x,\nabla y) = (b(x)+|\nabla y|)^{\alpha-2}\nabla y$$

with $\quad 0 < b_0 \leqq b(x) \leqq b_1 < +\infty$

EXAMPLE 2

$$a(x,\nabla y) = \frac{(b(x)+|\nabla y|)^{\alpha}}{c(x)+|\nabla y|^2}\nabla y$$

where $\quad 0 < b_0 \leqq b(x) \leqq b_1 < +\infty \, , \quad 0 < c_0 \leqq c(x) \leqq c_1 < +\infty \quad$ and

$$(\alpha-1)t^3-b_1t^2+c_0(\alpha+1)t+b_1c_0 \geqq 0 \qquad \text{for all } t>0$$

This operator can be found in Seidman (19)

2.- STUDY OF STATE EQUATION

Let us begin with a simple result (see for example Tolksdorf (20))

Lemma 2.1.- There are positive constants k_1 and k_2 depending only on N, α, Λ_1 and Λ_2 such that

a) $$\sum_{j=1}^{N} (a_j(x,\eta)-a_j(x,\eta'))\cdot(\eta_j-\eta'_j) \geq$$

$$\geq k_1 \cdot \begin{cases} (1+|\eta|+|\eta'|)^{\alpha-2}|\eta-\eta'|^2 & \text{if } \alpha\leq 2 \\ |\eta-\eta'|^{\alpha} & \text{if } \alpha\geq 2 \end{cases}$$

b) $$\sum_{j=1}^{N} |a_j(x,\eta)| \leq k_2(1+|\eta|)^{\alpha-2}|\eta|$$

Furthermore,

c) $$(a_0(x,s)-a_0(x,s'))(s-s') \geq \Lambda_3|s-s'|^2$$

d) $$|a_0(x,s)| \leq \Lambda_4|s|(1+|s|)^{\alpha-2}$$

for all $x\in\Omega$, all $s,s'\in\mathbb{R}$ and all $\eta,\eta'\in\mathbb{R}^N$

Lemma 2.2.- Assume $1<\alpha\leq 2$. Then

$$\int_\Omega (a(x,\nabla y)-a(x,\nabla y'))(\nabla y-\nabla y')\,dx \geq$$

$$\geq k_1||\nabla y-\nabla y'||^2_{L^\alpha(\Omega)}\,||1+|\nabla y|+|\nabla y'|\,||^{\alpha-2}_{L^\alpha(\Omega)}$$

Proof.- It is a simple consequence of a)-Lemma 2.1 and Hölder's inequality applied with $p=\frac{2}{\alpha}$ and $p'=\frac{2}{2-\alpha}$:

$$\int_\Omega |\nabla y-\nabla y'|^\alpha dx \leq$$

$$\leq \left(\int_\Omega |\nabla y-\nabla y'|^2(1+|\nabla y|+|\nabla y'|)^{\alpha-2}dx\right)^{\frac{\alpha}{2}}\left(\int_\Omega (1+|\nabla y|+|\nabla y'|)^\alpha dx\right)^{\frac{2-\alpha}{2}}$$

In the following result, we show continuous dependence respect to the data for this type of strongly nonlinear equations

Theorem 2.3.- Let $y\in W^{1,\alpha}(\Omega)$ be the weak solution of

$$\begin{cases} Ay = f & \text{in } \Omega \\ \dfrac{\partial y}{\partial n_A} = g & \text{on } \Gamma \end{cases}$$

and, for each $m\in N$, let $y_m\in W^{1,\alpha}(\Omega)$ satisfy

$$\begin{cases} Ay_m = f_m & \text{in } \Omega \\ \dfrac{\partial y_m}{\partial n_A} = g_m & \text{on } \Gamma \end{cases}$$

Assume that $f_m \longrightarrow f$ in $(W^{1,\alpha}(\Omega))'$ and $g_m \longrightarrow g$ in $W^{-\frac{1}{\beta},\beta}(\Gamma)$ as $m \longrightarrow \infty$. Then, $y_m \longrightarrow y$ in $W^{1,\alpha}(\Omega)$.

Proof.- From the relations satisfied by y and y_m, it follows that

$$\int_\Omega (a_0(x,y_m)-a_0(x,y))(y_m-y)\,dx + \int_\Omega (a(x,\nabla y_m)-a(x,\nabla y))(\nabla y_m-\nabla y)\,dx=$$

$$= (f_m - f, y_m - y)_\Omega + (g_m - g, y_m - y)_\Gamma .$$

Suppose $\alpha \geqq 2$. Applying a) and c) of lemma 2.1, we get

$$\Lambda_3 ||y_m - y||^2_{L^2(\Omega)} + k_1 ||\nabla y_m - \nabla y||^\alpha_{L^\alpha(\Omega)} \leq$$

$$\leq (||f_m - f||_{(W^{1,\alpha}(\Omega))'} + ||g_m - g||_{W^{-1/\beta,\beta}(\Gamma)}) ||y_m - y||_{W^{1,\alpha}(\Omega)}$$

Finally, considering in $W^{1,\alpha}(\Omega)$ the norm

$$||y|| = (|\int_\Omega y \, dx|^\alpha + ||\nabla y||^\alpha_{L^\alpha(\Omega)})^{1/\alpha}$$

equivalent to the usual thanks to Poincaré's inequality (see (12)) and using the hypotheses we obtain

$$y_m \longrightarrow y \qquad \text{in } W^{1,\alpha}(\Omega)$$

In the case $1<\alpha<2$, argumentation is similar using lemma 2.2.

In the rest of the section, we will assume $\underline{\alpha \geqq 2}$.

Given $y \varepsilon W^{1,\alpha}(\Omega)$, $y \neq 0$, let us consider the space

$$V^y(\Omega) = \{z \varepsilon H^1(\Omega) : \int_\Omega (1+|\nabla y|)^{\alpha-2} |\nabla z|^2 dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,y) z^2 dx < +\infty\}$$

endowed with the norm

$$||z||_{V^y(\Omega)} = (\int_\Omega (1+|\nabla y|)^{\alpha-2} |\nabla z|^2 dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,y) z^2 dx)^{1/2}$$

and $H^y(\Omega)$ the space completed of $C^\infty(\bar{\Omega})$ respect to the previous norm.

It may be easily verified that $V^y(\Omega)$ and $H^y(\Omega)$ are Hilbert spaces with the inner product

$$(z_1,z_2) = \int_\Omega (1+|\nabla y|)^{\alpha-2}\nabla z_1 \nabla z_2 \, dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,y)\, z_1 z_2 dx$$

Moreover, we have

$$W^{1,\alpha}(\Omega) \subset H^y(\Omega) \subset V^y(\Omega) \subset H^1(\Omega) \tag{2.1}$$

with continuous injections.

More general spaces of this type have been considered by Coffman et al. (7) and Trudinger (21),(22).

Theorem 2.4.- Let $F: L^2(\Gamma) \longrightarrow H^1(\Omega)$ be the functional defined by $F(u)=y_u$, where y_u is the weak solution of (1.8). Then F is Gâteaux differentiable when $H^1(\Omega)$ is endowed with the weak topology. Moreover, if $DF(u)v=z$, then z belongs to $H^{y_u}(\Omega)$ and it is the unique solution in this space of problem

$$\begin{cases} -\operatorname{div}\left(\frac{\partial a}{\partial \eta}(x,\nabla y_u)\nabla z\right) + \frac{\partial a_0}{\partial s}(x,y_u)\, z = 0 & \text{in } \Omega \\ \frac{\partial a}{\partial \eta}(x,\nabla y_u)\nabla z.\vec{n} = v & \text{on } \Gamma \end{cases} \tag{2.2}$$

Remark.- There exists a unique solution in $H^{y_u}(\Omega)$ of problem (2.2): it is enough to consider the bilinear form defined by

$$B(z_1,z_2) = \int_\Omega \nabla z_1^T \frac{\partial a}{\partial \eta}(x,\nabla y_u)\nabla z_2 dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,y_u)\, z_1 z_2 dx$$

which is a continuous and coercive form on $H^{y_u}(\Omega)$ thanks to (1.4), and then to apply the Lax-Milgram theorem.

Let us remind that the boundary condition of (2.2) can be interpreted in a rigorous manner (see (5)).

Proof.- Since the proof of this theorem is rather long, it will be convenient to divide it into some steps. For simplicity, we will write y instead of y_u.

Let be $u,v \in L^2(\Gamma)$ and $t>0$. We consider the following problems

$$\begin{cases} Ay_t = f & \text{in } \Omega \\ \dfrac{\partial y_t}{\partial \eta_A} = u+tv & \text{on } \Gamma \end{cases} \tag{2.3}$$

STEP 1

a) As $t \to 0$, $y_t \to y$ in $W^{1,\alpha}(\Omega)$

b) Sequence $\{\frac{y_t - y}{t}\}_{t>0}$ is bounded in $H^1(\Omega)$

c) $\int_\Omega (1+|\nabla y|)^{\alpha-2} |\frac{\nabla y_t - \nabla y}{t}|^2 \, dx \leq C \quad \forall t>0$

Proof of Step 1

a) It is a consequence of theorem 2.3

b) From the assumptions on y and y_t, we deduce

$$\int_\Omega (a_0(x,y_t)-a_0(x,y))(y_t-y)\,dx + \int_\Omega (a(x,\nabla y_t)-a(x,\nabla y))(\nabla y_t - \nabla y)\,dx =$$

$$= t.\int_\Gamma v(y_t-y)\,d\gamma \tag{2.4}$$

for all $t>0$.

Using mean value theorem and (1.2),(1.3), we obtain

$$(Ay_t-Ay,y_t-y) = \int_\Omega \frac{\partial a_0}{\partial s}(x,v_t)(y_t-y)^2\,dx + $$

$$+\int_\Omega (\nabla y_t-\nabla y)^T \frac{\partial a}{\partial \eta}(x,w_t)(\nabla y_t-\nabla y)\,dx \tag{2.5}$$

where $v_t=y(x)+\psi_t(x)(y_t(x)-y(x))$, $w_t(x)=\nabla y(x)+\theta_t(x)(\nabla y_t(x)-\nabla y(x))$ and $0<\psi_t(x),\theta_t(x)<1$.

Now, applying (1.4),(1.6), inequalities of Poincaré and Hölder, we get

$$C\,\|y_t-y\|^2_{H^1(\Omega)} \leq \Lambda_1\|\nabla y_t-\nabla y\|^2_{L^2(\Omega)}+\Lambda_3\|y_t-y\|^2_{L^2(\Omega)} \leq$$

$$\leq t\cdot\int_\Gamma v(y_t-y)\,d\gamma \leq t\|v\|_{L^2(\Gamma)}\|y_t-y\|_{H^1(\Omega)} \tag{2.6}$$

and so for all $t>0$ we have

$$\left\|\frac{y_t-y}{t}\right\|_{H^1(\Omega)} \leq \frac{1}{C}\|v\|_{L^2(\Gamma)} \tag{2.7}$$

c) In the sequel, we will denote $\frac{y_t-y}{t}$ by z_t. Thanks to (1.6) and a)-lemma 2.1 we have

$$(Ay_t-Ay,y_t-y) \geq k_1\|\nabla y_t-\nabla y\|^\alpha_{L^\alpha(\Omega)} \tag{2.8}$$

Therefore, it follows from (2.7) and (2.8) that

$$\frac{1}{t^2}\|\nabla y_t-\nabla y\|^\alpha_{L^\alpha(\Omega)} \leq \frac{1}{k_1 C}\|v\|^2_{L^2(\Gamma)} = C' \tag{2.9}$$

So, in virtue of (2.5),(1.4) and (2.9), we derive

$$\int_\Omega (1+|\nabla y|)^{\alpha-2}|\nabla z_t|^2\,dx \leq$$

$$\leq 2^{\alpha-2}\Big(\int_\Omega (1+|w_t|)^{\alpha-2}|\nabla z_t|^2 dx + \int_\Omega |\nabla y_t-\nabla y|^{\alpha-2}|\nabla z_t|^2 dx\Big) \leq$$

$$\leq 2^{\alpha-2}\Big(\frac{1}{\Lambda_1}\int_\Omega \nabla z_t^T\,\frac{\partial a}{\partial \eta}(x,w_t)\,\nabla z_t dx + \frac{1}{t^2}\int_\Omega |\nabla y_t-\nabla y|^{\alpha} dx\Big) \leq \tag{2.10}$$

$$\leq 2^{\alpha-2}\Big(\frac{1}{\Lambda_1}\int_\Gamma v\ z_t\ d\gamma + C'\Big) \leq C''$$

STEP 2

Let z be a limit point of $\{z_t\}_{t>0}$ in $H^1(\Omega)$ and $\{z_{t_n}\}_{n\in\mathbb{N}}$ a subsequence such that

$$z_{t_n} \longrightarrow z \quad \text{weakly in } H^1(\Omega) \text{ as } t_n \longrightarrow 0$$

Then, for each $\phi\in C^\infty(\bar{\Omega})$

$$\Big(\frac{Ay_{t_n}-Ay}{t_n},\phi\Big)_\Omega \longrightarrow \int_\Omega \nabla\phi^T\,\frac{\partial a}{\partial \eta}(x,\nabla y)\,\nabla z\,dx + \int_\Omega \phi\,\frac{\partial a_0}{\partial s}(x,y)\,z\,dx$$

Proof of Step 2

Let be $\phi\in C^\infty(\bar{\Omega})$ fixed. Once more, using mean value theorem we may deduce

$$\Big(\frac{Ay_{t_n}-Ay}{t_n},\phi\Big)_\Omega = \int_\Omega \nabla\phi^T\,\frac{\partial a}{\partial \eta}(x,u_{t_n})\,\nabla z_{t_n}\,dx +$$

$$+ \int_\Omega \phi\,\frac{\partial a_0}{\partial s}(x,v_{t_n})\,z_{t_n}\,dx = \int_\Gamma v\phi\,d\gamma \tag{2.11}$$

where $u_{t_n}(x) = \nabla y(x)+\omega_{t_n}(x)(\nabla y_{t_n}(x)-\nabla y(x))$, $0<\omega_{t_n}(x)<1$, ω_{t_n} depending on ϕ and v_{t_n} being as in (2.5).

By (2.9) and the conclusion c) of step 1, we have

$$\int_\Omega (1+|u_{t_n}|)^{\alpha-2}|\nabla z_{t_n}|^2 dx \leq$$

$$\leq 2^{\alpha-2}\left(\int_\Omega (1+|\nabla y|)^{\alpha-2}|\nabla z_{t_n}|^2 dx+\frac{1}{t_n^2}\|\nabla y_{t_n}-\nabla y\|^{\alpha}_{L^\alpha(\Omega)}\right) \leq C$$

On other hand, given $\vec{\psi}\varepsilon(D(\Omega))^N$

$$(1+|u_{t_n}|)^{\frac{\alpha-2}{2}}\vec{\psi} \longrightarrow (1+|\nabla y|)^{\frac{\alpha-2}{2}}\vec{\psi} \quad \text{in } (L^2(\Omega))^N$$

and moreover

$$\nabla z_{t_n} \longrightarrow \nabla z \quad \text{weakly in } (L^2(\Omega))^N$$

Therefore it follows that

$$\int_\Omega (1+|u_{t_n}|)^{\frac{\alpha-2}{2}}\vec{\psi}\nabla z_{t_n} dx \longrightarrow \int_\Omega (1+|\nabla y|)^{\frac{\alpha-2}{2}}\vec{\psi}\nabla z\, dx$$

and we may conclude that

$$(1+|u_{t_n}|)^{\frac{\alpha-2}{2}}\nabla z_{t_n} \longrightarrow (1+|\nabla y|)^{\frac{\alpha-2}{2}}\nabla z \quad \text{weakly in } (L^2(\Omega))^N \tag{2.12}$$

Now, let us consider the operator $G:(L^\alpha(\Omega))^N \longrightarrow (L^2(\Omega))^N$ defined by

$$G(\vec{g}) = \frac{\nabla\phi^T \frac{\partial a}{\partial \eta}(x,\vec{g})}{(1+|\vec{g}|)^{\frac{\alpha-2}{2}}}$$

Utilising (1.5) we deduce

$$|G(\vec{g})| \leq C|\nabla\phi|(1+|\vec{g}|)^{\frac{\alpha-2}{2}}$$

So, G is a superposition operator(M.A. Krasnoselskii et al. (11)), also called Nemyckiĭ operator (J. Nečas (18)), and thus G is continuos. Since $u_{t_n} \longrightarrow \nabla y$ in $(L^\alpha(\Omega))^N$ as $t_n \longrightarrow 0$, we have

$$\frac{\nabla\phi^T \frac{\partial a}{\partial \eta}(x,u_{t_n})}{(1+|u_{t_n}|)^{\frac{\alpha-2}{2}}} \longrightarrow \frac{\nabla\phi^T \frac{\partial a}{\partial \eta}(x,\nabla y)}{(1+|\nabla y|)^{\frac{\alpha-2}{2}}} \quad \text{in } (L^2(\Omega))^N \tag{2.13}$$

Combining (2.12) and (2.13), it follows that

$$\int_\Omega \nabla\phi^T \frac{\partial a}{\partial \eta}(x,u_{t_n})\nabla z_{t_n} dx \longrightarrow \int_\Omega \nabla\phi^T \frac{\partial a}{\partial \eta}(x,\nabla y)\nabla z\, dx \tag{2.14}$$

as $t_n \longrightarrow 0$.

We will complete the proof showing that

$$\int_\Omega \phi \frac{\partial a_0}{\partial s}(x,v_{t_n}) z_{t_n} dx \longrightarrow \int_\Omega \phi \frac{\partial a_0}{\partial s}(x,y)\, z\, dx \tag{2.15}$$

As in the preceding case, let us consider an operator $G_0 : L^\alpha(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$G_0(g) = \left(\frac{\partial a_0}{\partial s}(x,g)\right)^{1/2} \phi$$

Using hypothesis (1.6), we get

$$|G_0(g)| \leq (\Lambda_4)^{1/2}|\phi|(1+|g|)^{\frac{\alpha-2}{2}}$$

Therefore, G_0 is a superposition operator too. Since $v_{t_n} \longrightarrow y$ in $L^\alpha(\Omega)$ as $n \to \infty$ we have

$$\left(\frac{\partial a_0}{\partial s}(x,v_{t_n})\right)^{1/2} \phi \longrightarrow \left(\frac{\partial a_0}{\partial s}(x,y)\right)^{1/2} \phi \text{ in } L^2(\Omega) \tag{2.16}$$

On other hand, with the aid of the relations (2.4),(2.5) and (2.7) we obtain

$$\int_\Omega \frac{\partial a_0}{\partial s}(x,v_{t_n})|z_{t_n}|^2 dx \leq \|v\|_{L^2(\Gamma)} \|z_{t_n}\|_{H^1(\Omega)} \leq C'$$

Moreover, selecting a subsequence if necessary, we have that for almost every x in Ω

$$\left(\frac{\partial a_0}{\partial s}(x,v_{t_n}(x))\right)^{1/2} z_{t_n}(x) \longrightarrow \left(\frac{\partial a_0}{\partial s}(x,y(x))\right)^{1/2} z(x)$$

Then, we deduce that

$$\left(\frac{\partial a_0}{\partial s}(x,v_{t_n})\right)^{1/2} z_{t_n} \longrightarrow \left(\frac{\partial a_0}{\partial s}(x,y)\right)^{1/2} z \text{ weakly in } L^2(\Omega) \tag{2.17}$$

From (2.16) and (2.17), assertion (2.15) holds.

STEP 3

Let z be as in step 2. Then z belongs to $H^y(\Omega)$ and it is the unique solution in $H^{\gamma}(\Omega)$ of problem (2.2). Moreover $z_t \longrightarrow z$ weakly in $H^1(\Omega)$ as $t \to 0$

Proof of step 3

From (2.12) and (2.17), it follows that $z \in V^{\gamma}(\Omega)$. Furthermore, taking into account (2.11), the conclusion of previous step and the definition of $H^{\gamma}(\Omega)$ we get

$$\int_{\Omega} \nabla w^T \frac{\partial a}{\partial \eta}(x,\nabla y)\nabla z\, dx + \int_{\Omega} w\frac{\partial a_0}{\partial s}(x,y)\, z\, dx = \int_{\Gamma} vw\, d\gamma \quad \forall w \in H^{\gamma}(\Omega)$$

It remains to prove that $z \in H^{\gamma}(\Omega)$. For each $t_n > 0$ we consider the sequence

$$\{\Pi_m(z_{t_n})\}_{m\in\mathbb{N}} \subset W^{1,\alpha}(\Omega) \cap L^{\infty}(\Omega) \quad \text{where}$$

$$\Pi_m(t) = \begin{cases} t & \text{if } |t| \leq m \\ m & \text{if } t > m \\ -m & \text{if } t < -m \end{cases}$$

Thanks to (1.6),(2.10) and reminding (2.1), we have that $\{\Pi_m(z_{t_n})\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^{\gamma}(\Omega)$, for each $m \in \mathbb{N}$. Moreover, as $t_n \longrightarrow 0$

$$\Pi_m(z_{t_n}) \longrightarrow \Pi_m(z) \quad \text{weakly in } H^1(\Omega)$$

because Π_m is uniformly Lipschitz (see Fernández (8)). Consequently, as $t_n \longrightarrow 0$

$$\Pi_m(z_{t_n}) \longrightarrow \Pi_m(z) \quad \text{weakly in } H^{\gamma}(\Omega)$$

On other hand, it may be easily verified that

$$\| \Pi_m(z) \|_{V^Y(\Omega)} \leq \| z \|_{V^Y(\Omega)} \qquad \forall\ m \in \mathbb{N}$$

and then as $m \longrightarrow \infty$

$$\Pi_m(z) \longrightarrow z \quad \text{weakly in } H^Y(\Omega)$$

Remark.- We can prove directly that $z_t \longrightarrow z$ weakly in $H^Y(\Omega)$ in some situations (if $1 \leq N \leq 8$ or $\alpha > N$, for example) utilising the Hölder's inequality, b) and c) of step 1 and the Sobolev imbedding theorems (Adams (1)).

STEP 4

F is Gâteaux differentiable and $DF(u)v=z$

Proof of step 4

Continuing with the previous notation, if we consider the operator $T:L^2(\Gamma) \longrightarrow H^1(\Omega)$ defined by $T(v)=z$, it follows easily that T is linear and continuous: from (2.7) and the conclusion of step 3, we get

$$\| z \|_{H^1(\Omega)} \leq \liminf \| z_t \|_{H^1(\Omega)} \leq C' \| v \|_{L^2(\Gamma)}$$

Q.E.D.

3.- THE CONTROL PROBLEM (CASE $\alpha \geq 2$)

Let K be a non empty, convex and closed subset of $L^2(\Gamma)$ and let $J:L^2(\Gamma) \longrightarrow \mathbb{R}$ be the functional defined by

$$J(v) = \frac{1}{2} \int_\Omega |y_v - y_d|^2 dx + \frac{\nu}{2} \int_\Gamma |v|^2 d\gamma \tag{3.1}$$

where y_d is a fixed element of $L^2(\Omega)$ and ν is a non negative constant.

We consider the following control problem

$$(P_\alpha) \qquad \begin{array}{l} \text{Minimize } J(v) \\ \text{Subject to } v \in K \end{array}$$

As usual we derive existence of solution

Theorem 3.1.- Assume that

Either K is bounded in $L^2(\Gamma)$ or $\nu>0$. Then, there exists (at least) one solution of (P_α).

Proof.- Let $\{u_n\}_{n\in\mathbb{N}} \subset K$ be a minimizing sequence and $\{y_n\}_{n\in\mathbb{N}}$ the sequence of associated states. By the hypothesis, there exists $\bar{u} \in K$ and a subsequence $\{u_m\}_{m\in\mathbb{N}}$ such that

$$u_m \longrightarrow \bar{u} \quad \text{weakly in } L^2(\Gamma)$$

Let $\bar{y}$ be the associated state of $\bar{u}$. From theorem 2.3, we obtain that

$$y_m \longrightarrow \bar{y} \quad \text{in } W^{1,\alpha}(\Omega)$$

because $L^2(\Gamma) \subset W^{-\frac{1}{\beta},\beta}(\Gamma)$ with compact injection. The lower semicontinuity of J in the weak topology of $L^2(\Gamma)$ completes the proof.

Now, we get the first order optimality system by using theorem 2.4.

Theorem 3.2.- Let $\bar{u}$ be a solution of (P_α) and $\bar{y}$ the associated state. Then there exists a unique $\bar{p} \in H^{\bar{y}}(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(a(x,\nabla\bar{y}))+a_0(x,\bar{y}) = f & \text{in } \Omega \\ a(x,\nabla\bar{y})\vec{n} = \bar{u} & \text{on } \Gamma \end{cases} \tag{3.2}$$

$$\begin{cases} -\operatorname{div}((\frac{\partial a}{\partial \eta}(x,\nabla\bar{y}))^T \nabla\bar{p})+\frac{\partial a_0}{\partial s}(x,\bar{y})\bar{p} = \bar{y}-y_d & \text{in } \Omega \\ (\frac{\partial a}{\partial \eta}(x,\nabla\bar{y}))^T \nabla\bar{p}\,\vec{n} = 0 & \text{on } \Gamma \end{cases} \tag{3.3}$$

$$\int_\Gamma (\bar{p}+\nu\bar{u})(v.\bar{u})\,d\gamma \geqq 0 \quad \forall v \in K \tag{3.4}$$

Proof.- Since J is Gâteaux differentiable and K is convex we know that

$$J'(\bar{u})(v-\bar{u}) \geqq 0 \quad \forall v \in K$$

Continuing with the notation of theorem 2.4, and denoting $DF(\bar{u})(v-\bar{u})$ by z we have

$$J'(\bar{u})(v-\bar{u}) = \int_\Omega (\bar{y}-y_d)\,z\,dx + \nu\int_\Gamma \bar{u}(v-\bar{u})\,d\gamma$$

Let $\bar{p}$ be the unique solution of (3.3) in $H^{\bar{y}}(\Omega)$, thus

$$\int_\Omega (\bar{y}-y_d)\,z\,dx =$$

$$= \int_{\Omega} \nabla z^T \left(\frac{\partial a}{\partial \eta}(x,\nabla\bar{y})\right)^T \nabla\bar{p}\, dx + \int_{\Omega} z \frac{\partial a_0}{\partial s}(x,\bar{y})\,\bar{p}\, dx =$$

$$= \int_{\Omega} \nabla\bar{p}^{\,T} \left(\frac{\partial a}{\partial \eta}(x,\nabla\bar{y})\right) \nabla z\, dx + \int_{\Omega} \bar{p} \frac{\partial a_0}{\partial s}(x,\bar{y})\, z\, dx =$$

$$= \int_{\Gamma} \bar{p}(v-\bar{u})\, d\gamma$$

The last equality follows from theorem 2.4. Therefore

$$J'(\bar{u})(v-\bar{u}) = \int_{\Gamma} (\bar{p}+\nu\bar{u})(v-\bar{u})\, d\gamma \geq 0 \qquad \forall v \in K$$

Q.E.D.

4.- THE CONTROL PROBLEM (CASE $1<\alpha<2$)

In this section, we consider the same control problem (P_α) as in section 3 with slight variations. These are motivated by the facts that $W^{1,\alpha}(\Omega)$ is not included in $L^2(\Omega)$ and $L^2(\Gamma)$ is not included in $W^{-\frac{1}{\beta},\beta}(\Gamma)$ in some cases (for example, if $N=3$ and α is close to 1) see (1),(12),(17). Therefore, we formulate (P_α) in the following way

$$(P_\alpha) \quad \begin{cases} \text{Minimize } J(v) \\ \text{Subject to } v \in K \end{cases}$$

with K a non empty, convex and closed subset of $L^\beta(\Gamma)$ and $J:L^\beta(\Gamma) \longrightarrow R$ given by

$$J(v) = \frac{1}{\alpha}\int_\Omega |y_v - y_d|^\alpha \, dx + \frac{\nu}{\beta}\int_\Gamma |v|^\beta \, d\gamma \tag{4.1}$$

where y_d is a fixed element of $L^\alpha(\Omega)$ and $\nu \geq 0$.

As in the preceding case, it is easily followed

<u>Theorem 4.1</u>.- Assume that

Either K is bounded in $L^\beta(\Gamma)$ or $\nu>0$

Then there exists (at least) one solution of (P_α).

Let $\bar{u}$ be a solution of (P_α). The aim of this section is to prove that $\bar{u}$ satisfies an optimality system analogue to (3.2)-(3.4). Since we do not know if J is differentiable in case $\alpha<2$, we are going to consider a family of approximating problems that fall into the case $\alpha=2$. In order to do this, first let us consider the perturbed problem for each $\varepsilon>0$:

$$A_\varepsilon y = -\varepsilon \Delta y + Ay$$

So, A_ε satisfies the hypotheses (1.2)-(1.7) with $\alpha=2$. Hence, given $f\varepsilon(H^1(\Omega))'$ and $u\varepsilon H^{-1/2}(\Gamma)$ there exists a unique weak solution $y_\varepsilon(u)\varepsilon H^1(\Omega)$ of problem

$$\begin{cases} A_\varepsilon y = f & \text{in } \Omega \\ \dfrac{\partial y}{\partial n_{A_\varepsilon}} = u & \text{on } \Gamma \end{cases} \tag{4.2}$$

where

$$\frac{\partial y}{\partial n_{A_\varepsilon}}(x) = \sum_{j=1}^{N} (\varepsilon\frac{\partial y}{\partial x_j}(x) + a_j(x,\nabla y(x)))n_j(x)$$

Furthermore, we introduce the cost functional

$$J_\varepsilon(v) = \frac{1}{\alpha}\int_\Omega |y_\varepsilon(v)-y_d|^\alpha dx + \frac{\nu}{\beta}\int_\Gamma |v|^\beta d\gamma + \frac{1}{\beta}\int_\Gamma |v-\bar{u}|^\beta d\gamma$$

and the corresponding control problem

$$(P_\varepsilon) \quad \begin{array}{l} \text{Minimize } J_\varepsilon(v) \\ \\ \text{Subject to } v\varepsilon K \end{array}$$

Since J_ε is differentiable, we can derive the following result which is analogue to theorems 3.1 and 3.2

Theorem 4.2.- For each $\varepsilon>0$, there exists (at least) one solution of (P_ε). Moreover, if u_ε is a solution of (P_ε), then there exist elements y_ε and p_ε in $H^1(\Omega)$ such that

$$\begin{cases} -\text{div}(\varepsilon\nabla y_\varepsilon + a(x,\nabla y_\varepsilon)) + a_0(x,y_\varepsilon) = f & \text{in } \Omega \\ (\varepsilon\nabla y_\varepsilon + a(x,\nabla y_\varepsilon))\vec{n} = u_\varepsilon & \text{on } \Gamma \end{cases} \tag{4.3}$$

$$\left\{\begin{array}{l} -\operatorname{div}((\varepsilon I + \frac{\partial a}{\partial \eta}(x,\nabla y_\varepsilon))^T \nabla p_\varepsilon) + \frac{\partial a_0}{\partial s}(x,y_\varepsilon) p_\varepsilon = \\ = |y_\varepsilon - y_d|^{\alpha-2}(y_\varepsilon - y_d) \qquad \text{in } \Omega \\ \\ (\varepsilon I + \frac{\partial a}{\partial \eta}(x,\nabla y_\varepsilon))^T \nabla p_\varepsilon \vec{n} = 0 \qquad \text{on } \Gamma \end{array}\right. \tag{4.4}$$

$$\int_\Gamma (p_\varepsilon + \nu |u_\varepsilon|^{\beta-2} u_\varepsilon + |u_\varepsilon - \bar{u}|^{\beta-2}(u_\varepsilon - \bar{u}))(v - u_\varepsilon)\, d\gamma \geqq 0 \quad \forall v \varepsilon K \tag{4.5}$$

where I denotes the identity matrix NxN.

Our purpose is to pass to the limit in the optimality conditions of the perturbed problems, but first we need to prove two results.

Theorem 4.3.- Assume that $v_\varepsilon \longrightarrow u$ in $W^{-\frac{1}{\beta},\beta}(\Gamma)$ as $\varepsilon \rightarrow 0$. Then $y_\varepsilon(v_\varepsilon) \longrightarrow y_u$ in $W^{1,\alpha}(\Omega)$ as $\varepsilon \rightarrow 0$.

Proof.- For the sake of simplicity, we will write y_{v_ε} instead of $y_\varepsilon(v_\varepsilon)$. Applying lemma 2.2, Hölder's inequality and thanks to (1.6) we obtain

$$k_1 \| \nabla y_{v_\varepsilon} \|^2_{L^\alpha(\Omega)} \cdot \| 1 + |\nabla y_{v_\varepsilon}| \|^{\alpha-2}_{L^\alpha(\Omega)} + \Lambda_3 \| y_{v_\varepsilon} \|^2_{L^2(\Omega)} \leqq$$

$$\leqq \int_\Omega a(x,\nabla y_{v_\varepsilon}) \nabla y_{v_\varepsilon} dx + \int_\Omega a_0(x,y_{v_\varepsilon}) y_{v_\varepsilon} dx \leqq$$

$$\leqq (f, y_{v_\varepsilon})_\Omega + (v_\varepsilon, y_{v_\varepsilon})_\Gamma \leqq$$

$$\leqq \|f\|_{(W^{1,\alpha}(\Omega))'} \|y_{v_\varepsilon}\|_{W^{1,\alpha}(\Omega)} + \|v_\varepsilon\|_{W^{-\frac{1}{\beta},\beta}(\Gamma)} \|y_{v_\varepsilon}\|_{W^{1,\alpha}(\Omega)}$$

From here we derive that $\{y_{v_\varepsilon}\}_{\varepsilon>0}$ is bounded in $W^{1,\alpha}(\Omega)$. Thus, there exist a sequence $\varepsilon_n \to 0$ and $y \in W^{1,\alpha}(\Omega)$ such that

$$y_{v_{\varepsilon_n}} \longrightarrow y \quad \text{weakly in } W^{1,\alpha}(\Omega)$$

Furthermore $y_{v_{\varepsilon_n}}$ is the weak solution of (4.2) with $u=v_{\varepsilon_n}$ and then we have

$$\varepsilon_n \int_\Omega \nabla y_{v_{\varepsilon_n}} \nabla\psi dx + \int_\Omega a(x,\nabla y_{v_{\varepsilon_n}})\nabla\psi dx + \int_\Omega a_0(x,y_{v_{\varepsilon_n}})\psi dx =$$

$$= (f,\psi)_\Omega + (v_{\varepsilon_n},\psi)_\Gamma \text{ for all } \psi\in H^1(\Omega) \tag{4.6}$$

For proving that $y=y_u$ it is sufficient to pass to the limit in (4.6) as $\varepsilon_n \to 0$: Taking $\psi=y_{v_{\varepsilon_n}}$ in (4.6) it follows that

$$\varepsilon_n \| \nabla y_{v_{\varepsilon_n}} \|^2_{L^2(\Omega)} \leq (f,y_{v_{\varepsilon_n}})_\Omega + (v_{\varepsilon_n},y_{v_{\varepsilon_n}})_\Gamma \leqq C$$

for all $n\in\mathbb{N}$. Equivalently,

$$\varepsilon_n^{\frac{1}{2}} \| \nabla y_{v_{\varepsilon_n}} \|_{L^2(\Omega)} \leqq C' \quad \forall n\in\mathbb{N} \tag{4.7}$$

As a consequence, we get

$$\left| \varepsilon_n \int_\Omega \nabla y_{v_{\varepsilon_n}} \nabla\psi dx \right| \leq \varepsilon_n \| \nabla y_{v_{\varepsilon_n}} \|_{L^2(\Omega)} \| \nabla\psi \|_{L^2(\Omega)} \leqq$$

$$\leqq C' \varepsilon_n^{\frac{1}{2}} \| \nabla\psi \|_{L^2(\Omega)}$$

that tends to 0 as $\varepsilon_n \to 0$

On other hand, in virtue of b) and d) of lemma 2.1 we have

$$\int_\Omega |a(x,\nabla y_{v_{\varepsilon_n}})|^\beta dx \leq C\int_\Omega (1+|\nabla y_{v_{\varepsilon_n}}|)^\alpha dx \leq C'$$

$$\int_\Omega |a_0(x,y_{v_{\varepsilon_n}})|^\beta dx \leq \Lambda_4^\beta\int_\Omega (1+|y_{v_{\varepsilon_n}}|)^\alpha dx \leq C''$$

Therefore, we may infer that there exist a subsequence (again denoted $y_{v_{\varepsilon_n}}$), $\vec{\chi}\varepsilon(L^\beta(\Omega))^N$ and $\phi\varepsilon L^\beta(\Omega)$ such that

$$a(x,\nabla y_{v_{\varepsilon_n}}) \longrightarrow \vec{\chi} \quad \text{weakly in } (L^\beta(\Omega))^N$$

$$a_0(x,y_{v_{\varepsilon_n}}) \longrightarrow \phi \quad \text{weakly in } L^\beta(\Omega)$$

Let us introduce the element L of $(W^{1,\alpha}(\Omega))'$ defined by the formula

$$L(\psi) = \int_\Omega \vec{\chi}\nabla\psi dx + \int_\Omega \phi\psi dx$$

Letting ε_n tend to 0 in (4.6) we deduce

$$L(\psi) = (f,\psi)_\Omega + (u,\psi)_\Gamma \tag{4.8}$$

for all $\psi\varepsilon H^1(\Omega)$, and so for all $\psi\varepsilon W^{1,\alpha}(\Omega)$. Moreover, by (4.8)

$$\lim_{\varepsilon_n}\sup \int_\Omega a(x,\nabla y_{v_{\varepsilon_n}})\nabla y_{v_{\varepsilon_n}} dx + \int_\Omega a_0(x,y_{v_{\varepsilon_n}})y_{v_{\varepsilon_n}} dx \leq$$

$$\leq \lim_{\varepsilon_n}\sup \, ((f,y_{v_{\varepsilon_n}})_\Omega + (v_{\varepsilon_n},y_{v_{\varepsilon_n}})_\Gamma = (f,y)_\Omega + (u,y)_\Gamma =$$

$$= \int_\Omega \vec{\chi}\nabla y \, dx + \int_\Omega \phi y \, dx = L(y)$$

Since A is an operator that satisfies M-property (J.L. Lions (13,pg 173)) it is verified that

$$\int_\Omega a(x,\nabla y)\nabla\psi\, dx + \int_\Omega a_0(x,y)\psi\, dx = L(\psi) \qquad \forall\psi\in W^{1,\alpha}(\Omega)$$

and so, from (4.8) we get that $y=y_u$ and therefore

$$y_{v_\varepsilon} \longrightarrow y_u \quad \text{weakly in } W^{1,\alpha}(\Omega)$$

Finally, from the above results and lemma 2.2, we conclude

$$\lim_\varepsilon \sup k\cdot\| \nabla y_{v_\varepsilon} - \nabla y_u \|^2_{L^\alpha(\Omega)} \leq$$

$$\leq \lim_\varepsilon \sup \int_\Omega (a(x,\nabla y_{v_\varepsilon}) - a(x,\nabla y_u))(\nabla y_{v_\varepsilon} - \nabla y_u)\, dx \leq$$

$$\leq \lim_\varepsilon \sup \int_\Omega a(x,\nabla y_{v_\varepsilon})\nabla y_{v_\varepsilon}\, dx - \lim_\varepsilon \inf \int_\Omega a(x,\nabla y_{v_\varepsilon})\nabla y_u\, dx \leq$$

$$\leq \int_\Omega \vec{\chi}\nabla y_u\, dx - \int_\Omega \vec{\chi}\nabla y_u\, dx = 0$$

Hence, $y_{v_\varepsilon} \longrightarrow y_u$ strongly in $W^{1,\alpha}(\Omega)$ as $\varepsilon \to 0$.

Theorem 4.4.- Let u_ε be a solution of (P_ε). Set $\bar{y}=y_{\bar{u}}$ and $y_\varepsilon = y_\varepsilon(u_\varepsilon)$. Then, we have

$$u_\varepsilon \longrightarrow \bar{u} \quad \text{in } L^\beta(\Gamma) \tag{4.9}$$

$$y_\varepsilon \longrightarrow \bar{y} \quad \text{in } W^{1,\alpha}(\Omega) \tag{4.10}$$

$$J_\varepsilon(u_\varepsilon) \longrightarrow J(\bar{u}) \tag{4.11}$$

as $\varepsilon \to 0$.

Proof.- Applying previous theorem to $v_\varepsilon = \bar{u}$ $\forall\varepsilon>0$, we deduce

$$y_\varepsilon(\bar{u}) \longrightarrow \bar{y} \quad \text{in } W^{1,\alpha}(\Omega)$$

Since $\bar{u}\varepsilon K$, it follows that for all $\varepsilon>0$

$$\frac{1}{\beta}\| u_\varepsilon-\bar{u}\|^{\beta}_{L^\beta(\Gamma)} \leq J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\bar{u}) \leq C \tag{4.12}$$

Thus, $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\beta(\Gamma)$ and selecting a subsequence if necessary, we may infer that there exists $u\varepsilon K$ such that

$$u_\varepsilon \longrightarrow u \quad \text{weakly in } L^\beta(\Gamma)$$

Using once more theorem 4.3, we obtain

$$y_\varepsilon \longrightarrow y_u \quad \text{in } W^{1,\alpha}(\Omega)$$

From (4.12) and the lower semicontinuity of J_ε, we get

$$\frac{1}{\alpha}\| y_u-y_d\|^{\alpha}_{L^\alpha(\Omega)} + \frac{\nu}{\beta}\| u\|^{\beta}_{L^\beta(\Gamma)} + \frac{1}{\beta}\| u-\bar{u}\|^{\beta}_{L^\beta(\Gamma)} \leq$$

$$\leq \liminf J_\varepsilon(u_\varepsilon) \leq \liminf J_\varepsilon(\bar{u}) = \tag{4.13}$$

$$= \frac{1}{\alpha}\| \bar{y}-y_d\|^{\alpha}_{L^\alpha(\Omega)} + \frac{\nu}{\beta}\| \bar{u}\|^{\beta}_{L^\beta(\Gamma)} = J(\bar{u})$$

Since $\bar{u}$ is solution of (P_α), we know that $J(\bar{u})\leq J(u)$, thus $u=\bar{u}$ and

$$u_\varepsilon \longrightarrow \bar{u} \quad \text{weakly in } L^\beta(\Gamma)$$

$$y_\varepsilon \longrightarrow \bar{y} \quad \text{in } W^{1,\alpha}(\Omega) \qquad \text{and}$$

$$J_\varepsilon(u_\varepsilon) \longrightarrow J(\bar{u})$$

as $\varepsilon\to 0$.

Finally, we complete the proof noting that

$$0 \leq \limsup_{\varepsilon} \frac{1}{\beta} \| u_\varepsilon - \bar{u} \|^{\beta}_{L^\beta(\Gamma)} =$$

$$= \limsup_{\varepsilon} \left(J_\varepsilon(u_\varepsilon) - \frac{1}{\alpha} \| y_\varepsilon - y_d \|^{\alpha}_{L^\alpha(\Omega)} - \frac{\nu}{\beta} \| u_\varepsilon \|^{\beta}_{L^\beta(\Gamma)}\right) \leq$$

$$\leq J(\bar{u}) - \frac{1}{\alpha} \| \bar{y} - y_d \|^{\alpha}_{L^\alpha(\Omega)} - \frac{\nu}{\beta} \liminf_{\varepsilon} \| u_\varepsilon \|^{\beta}_{L^\beta(\Gamma)} \leq$$

$$\leq J(\bar{u}) - J(\bar{u}) = 0$$

Q.E.D.

Now we may pass to the limit in (4.3) to (4.5). First we note that using the same argument as in the proof of lemma 2.2 we get

$$\| \nabla p_\varepsilon \|^{2}_{L^\alpha(\Omega)} \leq C\left(\int_\Omega \frac{|\nabla p_\varepsilon|^2}{(1 + |\nabla y_\varepsilon|)^{2-\alpha}} dx\right) \tag{4.14}$$

On other hand, from hypothesis (1.6)

$$\| p_\varepsilon \|^{2}_{L^\alpha(\Omega)} \leq C' \int_\Omega \frac{\partial a_0}{\partial s}(x, y_\varepsilon) |p_\varepsilon|^2 dx \tag{4.15}$$

Combining (4.14), (4.15) and (4.4) we deduce

$$\| p_\varepsilon \|^{2}_{W^{1,\alpha}(\Omega)} \leq C''\left(\int_\Omega \frac{\partial a_0}{\partial s}(x, y_\varepsilon) |p_\varepsilon|^2 dx + \int_\Omega \nabla p_\varepsilon^T \left(\frac{\partial a}{\partial \eta}(x, \nabla y_\varepsilon)\right)^T \nabla p_\varepsilon dx\right) \leq$$

$$\leq C'' \int_\Omega |y_\varepsilon - y_d|^{\alpha-2}(y_\varepsilon - y_d) p_\varepsilon dx \leq C'' \| y_\varepsilon - y_d \|^{\alpha/\beta}_{L^\alpha(\Omega)} \| p_\varepsilon \|_{W^{1,\alpha}(\Omega)} \tag{4.16}$$

Thus, $\{p_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{1,\alpha}(\Omega)$ and we may conclude that there exist a sequence $\varepsilon_n \longrightarrow 0$ and $\bar{p} \varepsilon W^{1,\alpha}(\Omega)$ such that

$$p_{\varepsilon_n} \longrightarrow \bar{p} \quad \text{weakly in } W^{1,\alpha}(\Omega)$$

Applying (4.4) to p_{ε_n}, it follows

$$\varepsilon_n \| \nabla p_{\varepsilon_n} \|^2_{L^2(\Omega)} \leq \int_\Omega |y_{\varepsilon_n} - y_d|^{\alpha-2}(y_{\varepsilon_n} - y_d) p_{\varepsilon_n} dx \leq C$$

equivalently,

$$\varepsilon_n^{\frac{1}{2}} \| \nabla p_{\varepsilon_n} \|_{L^2(\Omega)} \leq C' \qquad \forall n \in \mathbb{N}$$

As a consequence, given $\psi \varepsilon C^\infty(\bar{\Omega})$ we have

$$\left| \varepsilon_n \int_\Omega \nabla\psi \nabla p_{\varepsilon n} dx \right| \leq \varepsilon_n \| \nabla p_{\varepsilon_n} \|_{L^2(\Omega)} \| \nabla\psi \|_{L^2(\Omega)} \leq C' \varepsilon_n^{\frac{1}{2}} \| \nabla\psi \|_{L^2(\Omega)}$$

that tends to 0 as $\varepsilon_n \longrightarrow 0$.

Furthermore, since $y_\varepsilon \longrightarrow \bar{y}$ in $W^{1,\alpha}(\Omega)$, we may suppose that

$$y_{\varepsilon_n}(x) \longrightarrow \bar{y}(x) \qquad \text{a.e. } x \varepsilon \Omega$$

$$\nabla y_{\varepsilon_n}(x) \longrightarrow \nabla\bar{y}(x) \qquad \text{a.e. } x \varepsilon \Omega$$

Thus, in virtue of (1.2) and (1.3), we get

$$\frac{\partial a_0}{\partial s}(x, y_{\varepsilon_n}(x)) \longrightarrow \frac{\partial a_0}{\partial s}(x, \bar{y}(x)) \quad \text{a.e. } x \varepsilon \Omega$$

$$\frac{\partial a}{\partial \eta}(x, \nabla y_{\varepsilon_n}(x)) \longrightarrow \frac{\partial a}{\partial \eta}(x, \nabla\bar{y}(x)) \quad \text{a.e. } x \varepsilon \Omega$$

Now, thanks to (1.5) and (1.6) it follows that

$$\left|\frac{\partial a_0}{\partial s}(x,y_{\varepsilon_n})\psi\right| \leq \Lambda_4(1+|y_{\varepsilon_n}|)^{\alpha-2}|\psi| \leq \Lambda_4|\psi|$$

$$\left|\frac{\partial a}{\partial \eta}(x,\nabla y_{\varepsilon_n})\nabla\psi\right| \leq C(1+|\nabla y_{\varepsilon_n}|)^{\alpha-2}|\nabla\psi| \leq C|\nabla\psi|$$

for all $n\varepsilon\mathbb{N}$.

By the Dominated Convergence Theorem we deduce

$$\frac{\partial a_0}{\partial s}(x,y_{\varepsilon_n})\psi \longrightarrow \frac{\partial a_0}{\partial s}(x,\bar{y})\psi \qquad \text{in } L^{\beta}(\Omega)$$

$$\frac{\partial a}{\partial \eta}(x,\nabla y_{\varepsilon_n})\nabla\psi \longrightarrow \frac{\partial a}{\partial \eta}(x,\nabla\bar{y})\nabla\psi \qquad \text{in } (L^{\beta}(\Omega))^N$$

as $\varepsilon_n \longrightarrow 0$ and therefore

$$\int_\Omega \nabla\psi^T \frac{\partial a}{\partial \eta}(x,\nabla\bar{y})^T\nabla\bar{p}\,dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,\bar{y})\psi\bar{p}\,dx =$$

$$= \int_\Omega |\bar{y}-y_d|^{\alpha-2}(\bar{y}-y_d)\psi\,dx \qquad \forall\psi\varepsilon C^\infty(\bar{\Omega})$$

Summarizing, we have

Theorem 4.5.- There exists $\bar{p}\varepsilon W^{1,\alpha}(\Omega)$ satisfying together with $\bar{u}$ and $\bar{y}$ the system

$$\begin{cases} -\mathrm{div}(a(x,\nabla\bar{y})) + a_0(x,\bar{y}) = f & \text{in } \Omega \\ a(x,\nabla\bar{y})\vec{n} = \bar{u} & \text{on } \Gamma \end{cases} \tag{4.17}$$

$$\begin{cases} -\mathrm{div}((\frac{\partial a}{\partial \eta}(x,\nabla\bar{y}))^T\nabla\bar{p}) + \frac{\partial a_0}{\partial s}(x,\bar{y})\bar{p} = |\bar{y}-y_d|^{\alpha-2}(\bar{y}-y_d) & \text{in } \Omega \\ (\frac{\partial a}{\partial \eta}(x,\nabla\bar{y}))^T\nabla\bar{p}\vec{n} = 0 & \text{on } \Gamma \end{cases} \tag{4.18}$$

$$\int_\Gamma (\bar{p} + \nu|\bar{u}|^{\beta-2}\,\bar{u})(v-\bar{u})\,d\gamma \geq 0 \qquad \forall v \in K \tag{4.19}$$

Moreover,

$$\int_\Omega \nabla\bar{p}^{\,T}\left(\frac{\partial a}{\partial \eta}(x,\nabla\bar{y})\right)^T \nabla\bar{p}\,dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,\bar{y})\,|\bar{p}|^2 dx \leq$$

$$\leq \int_\Omega |\bar{y}-y_d|^{\alpha-2}(\bar{y}-y_d)\,\bar{p}\,dx \tag{4.20}$$

It remains to prove the last inequality.

For the sake of simplicity, let us denote

$$M_\varepsilon(x) = \left(\frac{\partial a}{\partial \eta}(x,\nabla y_\varepsilon(x))\right)^T \text{ and } M(x) = \left(\frac{\partial a}{\partial \eta}(x,\nabla\bar{y}(x))\right)^T$$

First, we point out that

$$\xi^T M_\varepsilon(x)\xi = \xi^T M_\varepsilon^s(x)\xi \qquad \text{for all} \quad \xi \in \mathbb{R}^N$$

where $M_\varepsilon^s(x) = \frac{1}{2}(M_\varepsilon(x) + M_\varepsilon^T(x))$ is a symmetric and positive definite matrix (thanks to (1.4)). The same relation is valid for $M(x)$ and $M^s(x) = \frac{1}{2}(M(x) + M^T(x))$. Thus, applying the Cholesky Method to $M_\varepsilon^s(x)$ and $M^s(x)$ we deduce the existence of lower triangular matrices $L_\varepsilon(x)$ and $L(x)$ such that

$$M_\varepsilon^s(x) = L_\varepsilon(x)L_\varepsilon^T(x) \quad \text{and} \quad M^s(x) = L(x)L^T(x)$$

It follows from (4.16) that for all $\varepsilon>0$

$$\int_\Omega |L_\varepsilon \nabla p_\varepsilon|^2 dx = \int_\Omega \nabla p_\varepsilon^T M_\varepsilon^s \nabla p_\varepsilon dx = \int_\Omega \nabla p_\varepsilon^T M_\varepsilon \nabla p_\varepsilon dx \leq C$$

So, $\{L_\varepsilon \nabla p_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $(L^2(\Omega))^N$.

On other hand, taking into account (1.5) we have

$$\| L_\varepsilon \|_\rho = \| M_\varepsilon^s \|_\rho^{\frac{1}{2}} \leq \Lambda_2^{\frac{1}{2}}$$

where $\| \ \|_\rho$ denotes the spectral matrix norm (see Isaacson-Keller (9)). Consequently $\{L_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded, and furthermore thanks to (4.10) and (1.2)

$$L_\varepsilon(x) \longrightarrow L(x) \quad \text{a.e. } x\varepsilon\Omega$$

Then

$$L_\varepsilon \longrightarrow L \quad \text{in } (L^s(\Omega))^{N\text{x}N} \quad \text{as } \varepsilon \rightarrow 0$$

for all $s>1$, in particular for $s=\beta$

Since p_{ε_n} converges weakly to $\bar{p}$ in $W^{1,\alpha}(\Omega)$, we may deduce as in preceding cases that

$$L_{\varepsilon_n} \nabla p_{\varepsilon_n} \longrightarrow L\nabla\bar{p} \quad \text{weakly in } (L^2(\Omega))^N$$

as $\varepsilon_n \rightarrow 0$.

Finally, it is easily followed from (4.16),(4.10),(1.3) and (1.6) that

$$\left(\frac{\partial a_0}{\partial s}(x,y_{\varepsilon_n})\right)^{1/2} p_{\varepsilon_n} \rightarrow \left(\frac{\partial a_0}{\partial s}(x,\bar{y})\right)^{1/2} \bar{p} \quad \text{weakly in } L^2(\Omega)$$

Therefore, in virtue of the above results and (4.4), we conclude

$$\int_\Omega \nabla\bar{p}^{\,T} M \nabla\bar{p}\, dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,\bar{y})\, |\bar{p}|^2\, dx =$$

$$= \int_\Omega |L\nabla\bar{p}|^2\, dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,\bar{y})\, |\bar{p}|^2\, dx \leq$$

$$\leq \liminf \left(\int_\Omega |L_{\varepsilon_n} \nabla p_{\varepsilon_n}|^2\, dx + \int_\Omega \frac{\partial a_0}{\partial s}(x,y_{\varepsilon_n})\, |p_{\varepsilon_n}|^2\, dx\right) =$$

$$= \liminf \left(\int_\Omega \nabla p_{\varepsilon_n}^T M_{\varepsilon_n} \nabla p_{\varepsilon_n} dx + \int_\Omega \frac{\partial a_0}{\partial s}(x, y_{\varepsilon_n}) |p_{\varepsilon_n}|^2 dx \right) \leq$$

$$\leq \liminf \int_\Omega |y_{\varepsilon_n} - y_d|^{\alpha-2} (y_{\varepsilon_n} - y_d) p_{\varepsilon_n} dx =$$

$$= \int_\Omega |\bar{y} - y_d|^{\alpha-2} (\bar{y} - y_d) \bar{p} \, dx$$

Q.E.D.

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